$$\alpha_{k2} \equiv \frac{l^2}{2} \sum_{p,i,j} p^2 \alpha_{ij}^{\circ p} \Big[(\xi^i)^k - \sum_{n=1}^{N-1} \mu_n (\xi^n)^k \Big]$$

$$\rho_{kq1} \equiv \sum_{i,j} (\xi^i)^k \frac{\beta_{jk}}{\mu_j} \Big[\alpha_{ij}^{\circ} - \sum_p{}' \alpha_{ij}^{\circ p} \Big]$$

$$\rho_{kq2} \equiv l \sum_{p,i,j} p \alpha_{ij}^{\circ p} \frac{\beta_{jq}}{\mu_j} \Big[(\xi^i)^k - \sum_{n=1}^{N-1} \mu_n (\xi^n)^q \delta_{qk} \Big]$$

$$\rho_{kq3} \equiv \frac{l^2}{2} \sum_{p,j,i} p^2 \alpha_{ij}^{\circ p} \frac{\beta_{jq}}{\mu_j} \Big[(\xi^i)^k - \sum \mu_n (\xi^n)^q \delta_{qk} \Big]$$

Assumption that N denotes a set of identical particles or identical cells, does not lead to any significant simplification of Eqs. (2.13). Additional conditions will however be imposed on the constants of interaction appearing in these equations. In this case we shall have a continuous medium constructed with the help of macrocells. In the long wave approximation this medium will be described by equations of displacement of the center of mass of the cell and by equations of moments of various order. Increase in the number of particles in a macrocell will lead to the sharpening of the spectrum of the initial discrete system. If the macrocell coincides with the real cell of the discrete system, we note that we can draw conclusions from (2.13) concerning both, the acoustic and optical oscillations of the system at small k only. In order to make the spectrum more precise, at least two cells of the initial chain must be included into the macrocell.

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THE PROPAGATION OF WEAK DISCONTINUITIES IN THE SYSTEMS OF EQUATIONS OF MAGNETOGASDYNAMICS

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We consider the problem of weak discontinuities in quasi-linear hyperbolic systems and obtain transport equations for the case when the characteristic surfaces of the system have constant multiplicity. We also investigate weak discontinuities in magnetogasdynamics for the case when the characteristic surface is adjacent to a region of rest.

Authors of [1] deal with the problem of propagation of weak discontinuities in linear hyperbolic systems when the unknown functions of the system and their derivatives up to the n-th order remain continuous on the transition across the surface G, while the n-th order derivatives $(n \geqslant 1)$ suffer a finite discontinuity. Transport equation for quasi-linear systems when the magnitude of the jump in the values of the derivatives on transition across the surface of discontinuity is small, is obtained in [2]. Papers [3, 4] deal with the case of propagation of weak discontinuities for the equations of magnetogasdynamics, when the surface G is adjacent to a region of rest. The problem of weak discontinuities in quasi-linear hyperbolic systems $(n \geqslant 1)$ when the number of the independent variables is $m = 2$, is studied in detail in [5].

In the present paper we generalize the results of [5] to $m \geqslant 2$ and extend the results obtained in [1] for the characteristic surfaces of constant multiplicity, to the case of quasi-linear systems.

1. Let us consider an arbitrary, quasi-linear hyperbolic system of the form

$$L(\mathbf{U}) = \sum_{i=1}^{m} A^i \mathbf{U}_i + \mathbf{B} = 0 \left(\mathbf{U} = \begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix}, \quad \mathbf{U}_i = \frac{\partial \mathbf{U}}{\partial x_i} \right) \qquad (1.1)$$

Here A^i are matrices with elements $a_{lk}{}^i = a_{lk}{}^i\,(x_i, U)$, $(i = 1, \ldots, m)$ and $\mathbf{B}$ is a vector with elements $e_l = e_l\,(x_i, \mathbf{U})$.

We assume that on transition through the surface G whose equation is $\varphi(x_i) = 0$, function U and its derivatives up to the n-th order are continuous, while the leading n-th order derivative $U_{\varphi\ldots\varphi}$ has a finite discontinuity (the subscript $\varphi\ldots\varphi$ denotes n-fold differentiation) $[\mathbf{U}_{\varphi\ldots\varphi}] = \lim (\mathbf{U}^+_{\varphi\ldots\varphi} - \mathbf{U}^-_{\varphi\ldots\varphi})$ $(P^+ \to P,\ P^- \to P,\ P \in G)$

Here P^+ and P^- denote points lying on the opposite sides of the characteristic surface G. On introduction of new coordinates

$$x_i = \xi_i \ (i = 1, \ldots, m-1), \ \varphi(x_i) = \xi_m \quad (i = 1, \ldots, m)$$

the system (1.1) becomes

$$L^\varphi(\mathbf{U}) = A\mathbf{U}_\varphi + \sum_{i=1}^{m-1} A^i \mathbf{U}_i + \mathbf{B} = 0 \quad \left(A = \sum_{i=1}^{m} A^i \varphi_i \right) \qquad (1.2)$$

where A is its characteristic matrix. The jump $[\mathbf{U}_{\varphi\ldots\varphi}]$ satisfies the homogeneous system

$$A\,[\mathbf{U}_{\varphi\ldots\varphi}] = 0$$

consequently, if the rank of the matrix A is equal to $(n - s)$ then

$$[\mathbf{U}_{\varphi\ldots\varphi}] = \sum_{k=1}^{s} \sigma_k \mathbf{r}^k$$

Here σ_k are arbitrary scalars and $\mathbf{r}^k$ are linearly independent null vectors of the characteristic matrix $A\,(s \geqslant 1)$.

Let us now differentiate (1.2) n times with respect to $\varphi = \xi_m$ and perform a process analogous to that given in [1], ch. 5, taking into account the fact, that

$$\lim\ (f^+g^+ - f^-g^-) = [f][g] + [f]g^- + [g]f^+ \quad (P^+ \to P, P^- \to P)$$

This yields the following system of ordinary equations defining σ_k:

$$\sum_{k=1}^{s} \left\{ \alpha_k \sigma_k^{\cdot} + \sigma_k \mathbf{l}^\nu \left[\sum_{i=1}^{m-1} A^i \mathbf{r}_i{}^k + (\nabla_\mathbf{u} \mathbf{L}^\varphi(\mathbf{U})\, \mathbf{r}^k) + A_\varphi \mathbf{r}^k + \sum_{\nu=1}^{s} \sigma_\nu (\nabla_\mathbf{u} A \mathbf{r}^\nu)\, \mathbf{r}^k \right] \right\} = 0 \qquad (1.3)$$

for $n = 1$, and

$$\sum_{k=1}^{s}\left\{\alpha_k\sigma_k^{\bullet}+\sigma_k \mathbf{l}^j\left[\sum_{i=1}^{m-1} A^i\mathbf{r}_i^k+(\nabla_{\mathbf{u}}\mathbf{L}^{\varphi}(\mathbf{U})\,\mathbf{r}^k)+nA_{\varphi}\mathbf{r}^k\right]\right\}=0 \qquad (1.4)$$

$$(j=1,\ldots,s)$$

$$\mathbf{r}_i^k=\partial\mathbf{r}^k/\partial\xi_i$$

for $n>1$.

Here $\mathbf{l}^j$ denote linearly independent left null vectors of the characteristic matrix A, and α_k scalars. Here and in the following a dot ($^{\bullet}$) appearing in the superscript position will denote differentiation with respect to a parameter along the bicharacteristic ray.

When deriving (1.3) and (1.4), we have used a lemma concerning the bicharacteristic directions [1]. This lemma appears to be valid also for the quasi-linear systems, provided that the characteristic surfaces have constant multiplicity.

In the linear case, the transport equation is linear and intrinsic, since its coefficients depend only on x_i and φ_i [1]. This is not true for equations appearing in (1.3) and (1.4). In the linear case the coefficient of σ_k contains U_{φ}^{-}, while the equations appearing in (1.3) contain the products $\sigma_k\sigma_\nu$ and are therefore nonlinear. Systems (1.3) and (1.4) will yield the magnitude of the weak discontinuity only, when the value of the leading first order derivative is known on one of the sides of the surface. Nevertheless, these equations may be found useful in solving a number of applied problems, e.g. in investigating gas flows in magnetogasdynamics when the characteristic surface is adjacent to a region of rest, or to a region of uniform flow.

2. We shall use Eqs. (1.3) of Sect. 1 to investigate weak discontinuities (when $n=1$) which may appear in an unsteady flow of a perfect plasma. Equations of motion of the plasma have the form [6]

$$\frac{\partial\mathbf{H}}{\partial t}=\nabla\times(\mathbf{U}\times\mathbf{H}),\quad \frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{U})=0$$

$$\frac{\partial\mathbf{U}}{\partial t}+(\mathbf{U}\nabla)\mathbf{U}-\frac{\mu_l}{\rho}(\mathbf{H}\nabla)\mathbf{H}=-\frac{1}{\rho}\nabla\left(p+\frac{\mu_l}{2}H^2\right),\quad p=a\rho^{\gamma} \qquad (2.1)$$

Here $\mathbf{H}=\{h_1,h_2,h_3\}$ is the vector of magnetic intensity, $\mathbf{U}=\{u_1,u_2,u_3\}$ is the velocity vector, ρ is density, p is pressure, μ_l is magnetic permeability, a is a constant and γ is the adiabatic index.

Let the functions $\mathbf{H}$, $\mathbf{U}$ and ρ be continuous during the transition across the surface G defined by $\varphi(x_i)-t=0$, $(i=1,2,3)$ and their first derivatives undergo a finite discontinuity, then G is a characteristic surface satisfying the equation of the characteristics of the system (2.1)

$$\left(\frac{\mu_l}{\rho}H_n^2-1\right)\left[\frac{1}{\varphi^2}-\left(c^2+\frac{\mu_l}{\rho}H^2\right)+\frac{\mu_l}{\rho}c^2H_n^2\right]=0 \qquad (2.2)$$

$$H_n=\sum_{i=1}^{3}h_i\varphi_i,\quad \varphi=\left(\sum_{i=1}^{3}\varphi_i^2\right)^{1/2},\quad H^2=\sum_{i=1}^{3}h_i^2,\quad c=\left(\frac{\partial p}{\partial\rho}\right)^{1/2}$$

We begin with the case of magnetoacoustic waves, when

$$\frac{1}{\varphi^2}-\left(c^2+\frac{\mu_l}{\rho}H^2\right)+\frac{\mu_l}{\rho}c^2H_n^2=0 \qquad (2.3)$$

The order of the characteristic matrix of system (2.1) is equal to seven. Its rank is equal to six when (2.3) holds; this means that the matrix has one right and one left null vector. Formulas [6]

$$[\mathbf{H}_{\varphi}]=(H_n\mathbf{n}-\mathbf{H})\sigma,\quad [\mathbf{U}_{\varphi}]=\left(\frac{\mu_l}{\rho}H_n\mathbf{H}-\mathbf{n}\right)\sigma,$$

$$[\rho_{\varphi}]=\rho\left(\frac{\mu_l}{\rho}H_n^2-1\right)\sigma \qquad \mathbf{n}=\{n_i\},\quad n_i=\varphi_i/\varphi^2\quad (i=1,2,3)$$

define the jumps of the derivatives.

From now on, let us choose the time t as the parameter along the bicharacteristic ray; equations of the bicharacteristics of (2.1) become

$$\xi_i^{\cdot}=\left(\frac{\mu_l}{\rho}c^2h_iH_n-n_i\right)\left(\frac{\mu_l}{\rho}c^2H_n^2-\frac{1}{\varphi^2}\right)^{-1}, \quad \varphi_i^{\cdot}=0 \tag{2.4}$$

From this it clearly follows that $\varphi_i=\text{const}$ along the bicharacteristics and the bicharacteristics themselves will become straight lines in the case of adjoining to the region of rest ($\mathbf{U}=0$, $\mathbf{H}=\text{const}$, $\rho=\text{const}$) .

Inserting the appropriate values into the system (1.3), we obtain the following equation defining σ (in this case $s=1$)

$$2\left(\frac{\mu_l}{\rho}c^2H_n^2-\frac{1}{\varphi^2}\right)\sigma^{\cdot}+\sigma\left[-\frac{\Delta\varphi}{\varphi^4}+4\frac{\mu_l}{\rho}c^2H_n(\mathbf{n\Phi})+\frac{\mu_l}{\rho}c^2(\mathbf{H\Phi})\right]+\sigma^2L=0 \tag{2.5}$$

$$L=\left(\frac{\mu_l}{\rho}H_n^2-1\right)\left[3\left(c^2-\frac{1}{\varphi^2}\right)+\left(\frac{\mu_l}{\rho}c^2H_n^2-\frac{1}{\varphi^2}\right)+\left(\frac{\mu_l}{\rho}H_n^2-1\right)\left(\frac{1}{\varphi^2}+c^2(\gamma-1)\right)\right]=$$
$$=\text{const}$$

along the bicharacteristic

$$\mathbf{n}=\{n_i\},\quad \mathbf{\Phi}=\{\Phi_i\},\quad \Phi_i=\sum_{j=1}^{3}h_j\varphi_{ij}\quad (i=1,\ 2,\ 3)$$

In order to define $\sigma(t)$ from (2.5), we must express the coefficient accompanying σ in terms of t. Since $\varphi_i=\text{const}$ and h_i is constant along the bicharacteristic rays, it is sufficient to find $\varphi_{ij}=\varphi_{ij}(t)$ $(i,j=1,2,3)$. Equation (2.3) will be an identity with respect to $\xi_i (i=1,2,3)$. Let us differentiate it twice with respect to ξ_i . Taking (2.4) into account, we obtain the following system for φ_{ij} :

$$\begin{aligned}\dot\varphi_{11}&=\alpha_{11}\varphi_{11}^2+2\alpha_{12}\varphi_{11}\varphi_{12}+\alpha_{13}\varphi_{12}^2\\ \varphi_{12}&=\alpha_{12}\varphi_{12}^2+\alpha_{11}\varphi_{11}\varphi_{12}+\alpha_{13}\varphi_{12}\varphi_{22}+\alpha_{12}\varphi_{11}\varphi_{22}\\ \dot\varphi_{22}&=\alpha_{11}\varphi_{12}^2+2\alpha_{12}\varphi_{12}\varphi_{22}+\alpha_{13}\varphi_{22}^2\end{aligned} \tag{2.6}$$

$$\varphi_{i1}\xi_1^{\cdot}+\varphi_{i2}\dot\xi_2+\varphi_{i3}\dot\xi_3=0 \qquad (i=1,2,3) \tag{2.7}$$

$$a_{ik}=\frac{\delta_{ik}}{\varphi^4}+\frac{\xi_i^{\cdot}\xi_k^{\cdot}}{\xi_3^{\cdot}\varphi^4}-\frac{\mu_l}{\rho}c^2\left(4c^2\frac{\mu_l}{\rho}\varphi^2H_n^2-1\right)\left(h_i-h_3\frac{\xi_i^{\cdot}}{\xi_3^{\cdot}}\right)\left(h_k-h_3\frac{\xi_k^{\cdot}}{\xi_3^{\cdot}}\right)=\text{const}$$

where i, $k=1,2$ and δ_{ik} is the Kronecker delta.

First integrals

$$\frac{\alpha_{12}\varphi_{22}+\alpha_{11}\varphi_{12}}{\alpha_{12}\varphi_{11}+\alpha_{13}\varphi_{12}}=C_1,\qquad \frac{\alpha_{12}\varphi_{11}+\alpha_{13}\varphi_{12}}{\alpha_{12}(\varphi_{12}^2-\varphi_{11}\varphi_{22})}=C_2 \tag{2.8}$$

which are easily obtained for the system (2.6), enable us to reduce it to a single equation

$$\frac{d\varphi_{11}}{dt}=\frac{\alpha_{13}\Lambda+\alpha\Lambda^{1/2}}{2C_2^2\alpha_{12}^2}$$

$$\Lambda=[\alpha_{13}-C_2\varphi_{11}(\alpha_{11}-C_1\alpha_{13})]^2+4C_2\alpha_{12}^2\varphi_{11}(C_1C_2\varphi_{11}+1)$$
$$\alpha=2C_2\alpha_{12}^2\varphi_{11}+\alpha_{13}^2-C_2\alpha_{13}\varphi_{11}(\alpha_{11}-C_1\alpha_{13})$$

which, in turn, yields

$$\varphi_{11}=\frac{2k\{C_2k[2\alpha_{12}^2-\alpha_{13}(\alpha_{11}-C_1\alpha_{13})]-2(t+C_3)\alpha_{13}(\alpha_{11}\alpha_{13}-\alpha_{12}^2)\}}{4(t+C_3)^2(\alpha_{11}\alpha_{13}-\alpha_{12}^2)^2-C_2^2k^2[(\alpha_{11}-C_1\alpha_{13})^2+4\alpha_{12}^2C_1]}$$

$$k=\frac{\mu_l}{\rho}c^2H_n^2-\frac{1}{\varphi^2}$$

The remaining $\varphi_{ij}(t)$ are obtained from (2.7) and (2.8). Substituting $\varphi_{ij}(t)$ into (2.5), we finally obtain the transport equation for the system (2.1)

$$\sigma^{\cdot} + {}^1/_2 \left[\frac{1}{t+C} + \frac{1}{t+C_0}\right]\sigma + T\sigma^2 = 0 \qquad (2.9)$$

which agrees with the gas dynamical transport equation [4] with accuracy to the constant terms.

From (2. 9) we obtain $\sigma = \dfrac{1}{\sqrt{t+C}\,\sqrt{t+C_0}\,[M + 2T \ln|\sqrt{t+C} + \sqrt{t+C_0}|]}$

Here M, C and C_0 are arbitrary constants and $T = L/k$, i. e. the law of variation of $\sigma(t)$ along the bicharacteristics coincide with that appearing in the standard gas dynamics [4]. When

$$\frac{\mu_l}{\rho} H_n^2 - 1 = 0 \qquad (2.10)$$

and if we assume that $\mathbf{H}$ and $\mathbf{n}$ are not parallel, then the characteristic matrix again has one right and one left null vector and the jumps in the values of the derivatives are given by

$$[\mathbf{H}_\varphi] = H_n \varphi^2 (\mathbf{H} \times \mathbf{n})\,\sigma, \quad [\mathbf{U}_\varphi] = \varphi^2 (\mathbf{H} \times \mathbf{n})\,\sigma, \quad [\rho_\varphi] = 0 \qquad (2.11)$$

On substituting the relevant quantities into (1. 3) we find, that the coefficients of σ and σ^2 are equal to zero and the transport equation becomes $\sigma^{\cdot} = 0$, i. e. $\sigma =$ const along a bicharacteristic. Weak discontinuities which have occurred at the initial instant retain their constant intensity.

Note. If the characteristic surfaces have constant multiplicity [1], weak discontinuities propagate along the bicharacteristic rays. However, when the characteristic surfaces have variable multiplicity [1], no general theory exists for either the linear or the quasi-linear systems. Investigation of the weak discontinuities in magnetogasdynamics, shows that for variable multiplicity of the characteristic surfaces in some cases (e. g. when (2.10) holds, $\mathbf{H}$ is parallel to $\mathbf{n}$, $c^2\varphi_i = (\mu_l/\rho) h_i H_n$), a system of ordinary equations defining $\sigma_k(t)$ is unobtainable. In other cases weak discontinuities propagate along the bicharacteristics; in particular if (2. 10) holds, $\mathbf{H}$ is parallel to $\mathbf{n}$ and $c^2\varphi_i \neq (\mu_l/\rho) h_i H_n$, the jumps are given by

$$\left[\frac{\partial u_i}{\partial \varphi}\right] = -\varphi_1 \sigma_i \quad (i = 2, 3)$$

$$\sum_{j=1}^{3} \varphi_j \left[\frac{\partial u_i}{\partial \Psi}\right] = 0 \quad [\mathbf{H}_\varphi] = -H_n [\mathbf{U}_\varphi], \quad [\rho_\varphi] = 0$$

and $\sigma_1(t)$ and $\sigma_2(t)$ can be found from the system $\sigma_1^{\cdot} = 0$, $\sigma_2^{\cdot} = 0$, i. e. in this case weak discontinuities retain constant intensity.

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